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On the flux of heat through laminar wavy liquid layers

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Dynamically passive transfer of heat across a layer of liquid supporting a progressive, periodic surface wave is approached from a Lagrangian viewpoint. The model layer considered is the region between two constant pressure surfaces of a Gerstner wave and the thermal boundary conditions are that the average temperature of any surface particle remains constant and that there is horizontal homogeneity of the average temperature field.

It is shown that fluctuations in the temperature of any particle are negligibly small for ordinary liquids and a uniformly valid approximation to the average temperature of each particle is presented.

The extent to which the flux of heat through the layer is augmented is computed for typical cases and it is shown to be at most doubled. Indication is given of extensions of the method to other kinds of progressive waves and to situations in which the boundary conditions are unsteady and spatially inhomogeneous.

1. Introduction

Surface distortions of the progressive wave type on a layer of fluid can be expected to effect the flux of heat or any other scalar quantity through the layer in at least two ways. First, conductive transfer is sensitive to the geometry of the layer, and second, there will be some convective transfer in the body of the layer. This investigation presents a method of accurately estimating such fluxes.

The practical significance of the problem has long been recognized by those interested in heat transfer estimates. For example, McAdams (1954) apparently recommends as much as a 20 % increase in heat flux through condensing films on vertical tubes when the film shows wavy characteristics as compared to the same film in a non-wavy state.

For the sake of definiteness we restrict our terminology to the transfer of heat and we consider a specific boundary value problem in two space dimensions. An important simplification that we adopt is to neglect both buoyancy induced motion and property variation with temperature. That is, the temperature field T is taken to be dynamically passive. It has no effect on the velocity field u_i of the progressive wave, which must therefore be part of the given data of the problem. Even with this assumption and the neglect of dissipative heat generation the solution of the energy equation is not a trivial one, as we shall show, because of the complexity of the velocity field associated with progressive waves. Mass transfer across the bounding surfaces of the fluid is not permitted and thus the boundaries will be material surfaces. It is therefore natural to consider a Lagrangian formulation of the heat transfer problem, and for the sake of studying the simplest situation the thermal boundary conditions will be that the average temperature of each particle on the upper surface is a constant T_1, and the average temperature of each particle on the lower surface is also a constant, T_2 ($T_1 \neq T_2$).

One simple description, in Lagrangian form, of the kinematics of a layer supporting a finite amplitude progressive wave is the Gerstner solution to the incompressible, inviscid, hydrodynamic equations and we will adopt this particular wave solution to describe the fluid layer. The analytic description of the Gerstner wave is presented in § 2 and a graphical description of one wavelength of the wave is displayed in figure 1. The wave is periodic and is progressing with a velocity c. The fully inked-in lines of constant β in figure 1 are samples of constant pressure surfaces in the solution. β is a non-dimensional Lagrangian co-ordinate to be defined in § 2. We point out here that we consider any region between two constant pressure surfaces to be a possible layer to which our solution will apply.

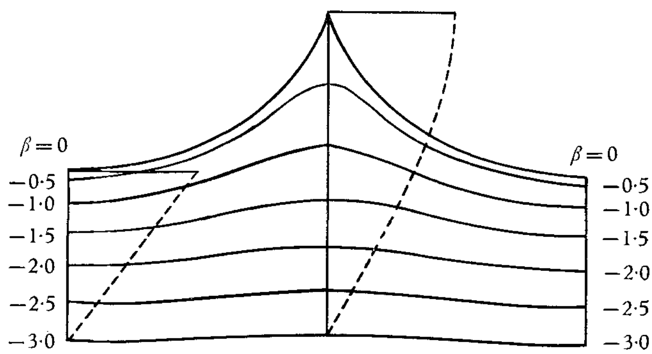

FIGURE 1. The Gerstner wave. – – –, normalized temperature profiles $(T^0(\beta) - T_2)/(T_1 - T_2)$.

As far as the transport of heat is concerned, the Gerstner wave is expected to be kinematically typical of those progressive waves in which there is little particle drift and for which the particle orbits grow larger monotonically as one approaches the upper surface. In fact it has been shown elsewhere (O'Brien 1965), by adopting the solution procedure reported in this paper to the case of a shallow gravity wave, that the temperature profile for a Gerstner wave is indistinguishable from that for a shallow gravity wave under the same thermal boundary conditions provided that the ratios of amplitude-to-depth and amplitude-to-wavelength are the same for both waves.

2. Energy equation for the Gerstner wave

The Gerstner wave is an exact solution of the inviscid, incompressible, hydrodynamic equations presented in terms of Lagrangian variables a and b. The trajectory of any particle whose material co-ordinates are a and b is given by Lamb (1945) as

$$x(a, b, t) = a + k^{-1} e^{kb} \sin k(a + ct), \quad y(a, b, t) = b - k^{-1} e^{kb} \cos k(a + ct), \tag{2.1}$$

where x and y are Eulerian position co-ordinates, c is the wave speed, $c = g^{\frac{1}{2}} k^{\frac{1}{2}}$, and k the wave-number. Thus a physical interpretation of (a, b) is that it is the centre of the circle in which the particle moves.

The Jacobian of the transformation, $J = [\partial(x,y)]/[\partial(a,b)]$, is $1-e^{2kb}$ from which it is seen that b can have all values less than zero. The Gerstner wave has considerable analytical advantages over other kinds of wave solutions; its Jacobian is time independent and, within the approximation to be developed in §3, a closed form solution to the temperature field can be obtained for all amplitudes, wavelengths and depths that Gerstner waves can exhibit.

When (2.1) is written in non-dimensional form by means of the transformations $\alpha = ka$, $\beta = kb$ and $\tau = kct$, the energy equation, which has the Eulerian form $(\partial T/\partial t) + u_i(\partial T/\partial x_i) = D\Delta^2 T$, becomes

$$\begin{aligned} T_\tau(\alpha,\beta,\tau) = {} & c^{-1}DkJ^{-2}\{[1+e^{2\beta}-2e^{\beta}\cos(\alpha+\tau)]T_{\alpha\alpha} \\ & +[1+e^{2\beta}+2e^{\beta}\cos(\alpha+\tau)]T_{\beta\beta}-4e^{\beta}\sin(\alpha+\tau)T_{\alpha\beta} \\ & -4e^{2\beta}(1-e^{2\beta})^{-1}\sin(\alpha+\tau)T_\alpha+4e^{2\beta}(1-e^{2\beta})^{-1}[1+e^{\beta}\cos(\alpha+\tau)]T_\beta\}, \end{aligned} \tag{2.2}$$

where subscripts refer to partial derivatives. For example

$$T_{\alpha\beta} = \partial^2 T/\partial\alpha\,\partial\beta.$$

The boundary conditions are

$$\overline{T}(\beta_1) = T_1 \quad \text{a constant}, \tag{2.3}$$

$$\overline{T}(\beta_2) = T_2 \quad \text{a constant}, \tag{2.4}$$

where $\beta = \beta_1$ is the upper surface, $\beta = \beta_2$ is the lower surface, and $\overline{T}$ means the average temperature, the average to be taken over either a wavelength or a wave period.

(2.2) is a linear partial differential equation with coefficients that depend on all three independent variables. The Eulerian form of the energy equation is of course of just the same type but the Lagrangian form has several advantages. The boundary conditions for the quasi-steady situation are of the simplest kind and more significantly a natural perturbation analysis suggests itself from this viewpoint. The perturbation scheme makes use of the fact that the parameter $(Dk)/c$ is extremely small for ordinary liquids except for waves of very large wave-number. In fact if surface tension is to be included this parameter would be small for all wave-lengths since $c(k)$ then exhibits a minimum. The smallness of this ratio suggests the fact that in the time-scale of one period the diffusive loss or gain of temperature by a particle must be small. The diffusion of heat in water at room temperature $(Dk)/c$, which will be denoted by ϵ for the remainder of the paper, can be computed to be no greater than 10^{-4}. For such a case it is reasonable to expect that the quasi-steady solution—that is, the one in which each particle exhibits a steady average temperature—will also be one in which the temperature of any particular particle is effectively constant during the motion.

In the following section a perturbation analysis is developed and it is proved that the fluctuations in the temperature of a particle are indeed of order ϵ.

Furthermore a zeroth order solution to the temperature field is constructed which is a uniformly valid approximation to the average temperature of each particle. The error is again of order ϵ.

3. The zeroth-order approximation

If ϵ in (2.2) is set equal to zero any time-independent temperature field will be a solution of the equation. This is just a reflexion of the physical consideration that loss or gain or thermal energy to a particle in our model can only be by molecular diffusion. When this mechanism is removed every particle retains a constant temperature no matter how it is transported by the velocity field. Thus the proper goal of this investigation is to find a zeroth-order solution which is a uniformly valid approximation to the limit as ϵ approaches zero of the solution to the full equation (2.2). This can be achieved in the following way.

The temperature field is expanded in the form

$$T(\alpha,\beta,\tau) = \sum_{n=0}^{\infty} f_n(\beta)\cos n(\alpha+\tau) + \sum_{n=0}^{\infty} g_n(\beta)\sin n(\alpha+\tau), \tag{3.1}$$

where we have used the fact that in the quasi-steady case T must be a periodic function of $(\alpha+\tau)$. Substituting this series form into the governing equation (2.2) and collecting coefficients of the same harmonic function we find the general terms are of the form

$$g_n(\beta) = -\epsilon\{L_1^{(n)}f_n + L_2^{(n)}f_{n-1} + L_3^{(n)}f_{n+1}\}, \quad (n \geqslant 1); \tag{3.2}$$

$$f_n(\beta) = \epsilon\{L_1^{(n)}g_n + L_2^{(n)}g_{n-1} + L_3^{(n)}g_{n+1}\}, \quad (n \geqslant 1); \tag{3.3}$$

where $$L_1^{(n)} = (1-e^{2\beta})^{-3}\{(1-e^{4\beta})([d^2/d\beta^2]-n^2) + 4e^{2\beta}(d/d\beta)\},$$

$$L_2^{(n)} = e^{\beta}(1-e^{2\beta})^{-2}\{-([d^2/d\beta^2]+[n-1]^2) + 2(n-1)([d/d\beta]+e^{2\beta}[1-e^{2\beta}]^{-1}) - 2e^{2\beta}(1-e^{2\beta})^{-1}[d/d\beta]\},$$

and

$$L_3^{(n)} = e^{\beta}(1-e^{2\beta})^{-2}\{-([d^2/d\beta^2]+[n+1]^2) + 2(n+1)([d/d\beta]+e^{2\beta}[1-e^{2\beta}]^{-1}) - 2e^{2\beta}(1-e^{2\beta})^{-1}[d/d\beta]\}.$$

Also $$g_0 = 0, \tag{3.4}$$

and f_0 satisfies $$L_1^{(0)}f_0 = -L_3^{(0)}f_1. \tag{3.5}$$

In particular from (3.2), (3.3) and (3·4), g_1 and f_1 satisfy the following equations:

$$g_1 = -\epsilon\{L_1^{(1)}f_1 + L_2^{(1)}f_0 + L_3^{(1)}f_2\}, \tag{3.6}$$

$$f_1 = \epsilon\{L_1^{(1)}g_1 + L_3^{(1)}g_2\}. \tag{3.7}$$

By taking the mean of (3.1) we deduce from (2.3) and (2.4) the following boundary conditions:

$$f_0(\beta_1) = T_1, \quad f_0(\beta_2) = T_2. \tag{3.8}$$

The operators $L_{1,2,3}^{(n)}$ are independent of ϵ and are well behaved except at $\beta = 0$. The $\beta = 0$ surface exhibits a physically unrealistic cusp as can be seen from figure 1 and we exclude it from consideration as a possible upper surface of the layer. For $\beta < 0$, since f_n and g_n are at most finite for all n, it follows from (3.6) and (3.7) that

$$f_1 = O(\epsilon), \quad g_1 = O(\epsilon). \tag{3.9}$$

Also from (3.2), (3.3) and (3.9) we have

$$f_n = O[(n-1)^2\epsilon^n], \quad g_n = O[(n-1)^2\epsilon^n], \quad (n > 1). \tag{3.10}$$

It can also be demonstrated that

$$\sum_{n=1}^{\infty}(n-1)^2\epsilon^n < \epsilon\left[1+\frac{1}{|\ln\epsilon|}+\frac{2}{|\ln\epsilon|^2}+\frac{2}{|\ln\epsilon|^3}\right], \tag{3.11}$$

which, when combined with (3.10), shows that the Fourier series (3.1) is also a possible perturbation series and that in the limit as ϵ approaches zero, summation of the time dependent terms of the series are of order ϵ.

Clearly, therefore, $f_0(\beta)$ is the zeroth order solution we are seeking. From (3.1) it is evident that $f_0(\beta)$ is the average temperature of any particle which lies on a constant β surface and the physical meaning of (3.11) is that the fluctuation in particle temperature about its average value is $O(\epsilon)$ as $\epsilon \to 0$, a result which could have been anticipated on physical grounds. But, as we show in the following paragraph the preceding construction also suggests an approximation to $f_0(\beta)$ which is accurate to $O(\epsilon)$.

Temperature profile

In (3.5) the right-hand side is $O(\epsilon)$ and writing $T^0(\beta)$ for the zeroth approximation to $f_0(\beta)$ we have from (3.5) and (3.8)

$$L_1^{(0)}T^0(\beta) = 0, \quad T^0(\beta_1) = T_1, \quad T^0(\beta_2) = T_2. \tag{3.12}$$

Integration is straightforward and yields as a solution

$$\frac{T_1 - T^0(\beta)}{T_1 - T_2} = \frac{\ln(\cosh\beta_1/\cosh\beta)}{\ln(\cosh\beta_1/\cosh\beta_2)} \tag{3.13}$$

and

$$T^0_\beta = -\frac{T_1 - T_2}{\ln(\cosh\beta_1/\cosh\beta_2)}\frac{(1-e^{2\beta})}{(1+e^{2\beta})}. \tag{3.14}$$

If further terms in the approximation series are calculated in the fashion suggested in §4 it is found that the error in approximating $f_0(\beta)$ to $T^0(\beta)$ is of order ϵ^2 uniformly, including the bounding surfaces, provided that, as was mentioned previously, $\beta = 0$ is excluded as a possible surface. The basic solution (3.13) also satisfies the boundary condition (3.8) identically so that the problem as posed here does not give rise to a singular perturbation. The surface layers are in fact indistinguishable from any other constant β surface in the interior of the fluid. If boundary conditions are imposed which alter this fact, for example if specified fluctuations in temperature are demanded of surface particles, then presumably some type of surface thermal boundary layer may be expected but its significance remains to be explored and is outside the scope of this paper.

The solution (3.13) is displayed in figure 2 for the special case $\beta_1 \to 0$, $\beta_2 = -3$ and its Eulerian counterpart is indicated by the temperature profiles superimposed on the Gerstner wave of figure 1. It is of some interest to determine to what extent this solution differs from that of pure conduction through a stationary layer of the same geometry. To determine such a solution we could again employ the transformation (2.1) with $c = 0$. The steady temperature profile would then be one that satisfies the right-hand side of (2.2) with $\tau = 0$. It is immediately evident that the solution will be a function of α as well as β or in other words constant β surfaces are no longer isothermal surfaces. We have carried out an analogue solution of this steady-state conduction problem using conducting paper and a comparison of the isothermal surfaces so obtained with constant β surfaces is presented in figure 3.

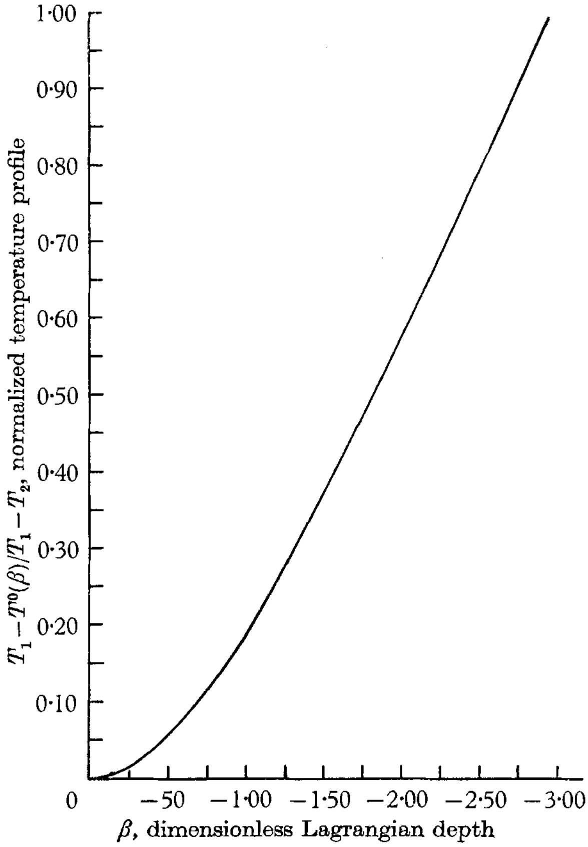

FIGURE 2. Temperature profile for Gerstner wave: ($\beta_1 \to 0$, $\beta_2 = -3$).

Evaluation of heat flux

Since $\beta = \beta_1$ is a free surface the instantaneous heat flux Q through a wavelength of surface of unit width is given by the integral

$$Q = D\int_0^{2(\pi/k)} \nabla T\eta_i(\partial x_i/\partial a)\,da,$$

where η_i is the unit vector normal to the surface, x_i is the Eulerian position vector of the surface, and all quantities are to be evaluated at $\beta = \beta_1$.

We find after some manipulations (O'Brien 1965) that the heat flux corresponding to the temperature field (3.13) is given by

$$Q = -2\pi D(T_1 - T_2)\{\ln(\cosh\beta_1/\cosh\beta_2)\}^{-1}. \tag{3.15}$$

It is useful to compare such a heat flux with the flux one would observe through an equivalent slab; that is, a quiescent film obtained by allowing the wavy film to come to rest. If $b = b_1$ is the wave surface then the height of the equivalent quiescent surface is given by $y = y_1$, where $b_1 - y_1 = (2k)^{-1}e^{2kb_1}$. If Δy is the dimensionless equivalent slab depth corresponding to the surfaces $\beta = \beta_1$ and $\beta = \beta_2$ we have

$$\Delta y = k(y_1 - y_2) = \beta_1 - \beta_2 - \tfrac{1}{2}(e^{2\beta_1} - e^{2\beta_2}).$$

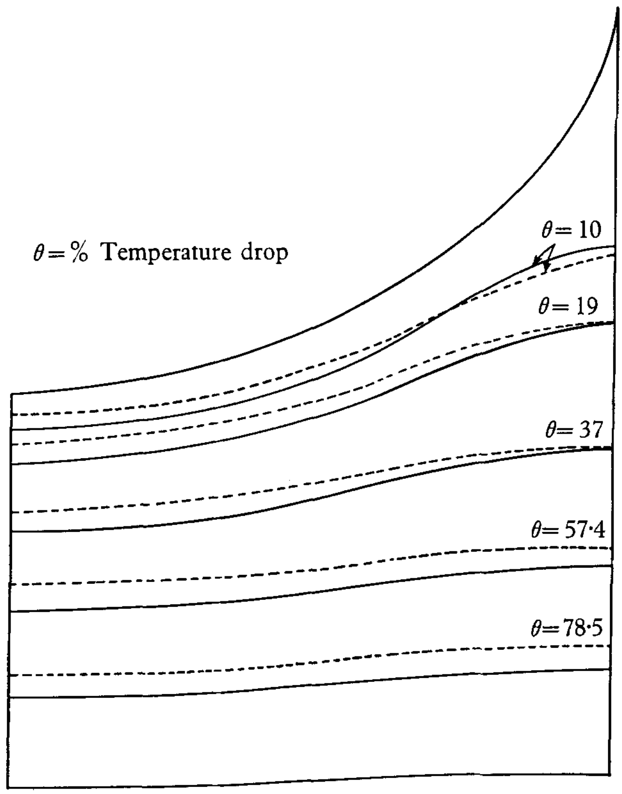

FIGURE 3. Comparison of conduction solution and $T^0(\beta)$: ($\beta \to 0$, $\beta_2 = -3$). ---, isotherms of conduction solution; —, $T^0(\beta)$.

The heat flux/unit wavelength/unit width through such an equivalent slab is given by

$$Q_{\text{slab}} = 2\pi D(T_1 - T_2)(\Delta y)^{-1}$$

and thus finally

$$Q/Q_{\text{slab}} = [(\beta_2 - \beta_1) + \tfrac{1}{2}(e^{2\beta_1} - e^{2\beta_2})]/[\ln(\cosh\beta_1/\cosh\beta_2)]. \tag{3.16}$$

Some numerical results for large-amplitude waves

Consider the maximum amplitude wave which occurs when β_1 approaches zero. Then

$$Q/Q_{\text{slab}} \to (\beta_2 + \tfrac{1}{2}[1 - e^{2\beta_2}])/(-\ln\cosh\beta_2).$$

For $\beta_2 = -1$ or $b = \lambda/2\pi$, where λ is the wavelength, (3.15) predicts a heat flux ratio of 1.31,

$$\text{for} \quad \beta_2 = -2, \quad Q/Q_{\text{slab}} = 1{\cdot}15,$$

and

$$\text{for} \quad \beta_2 = -3, \quad Q/Q_{\text{slab}} = 1{\cdot}08.$$

When $\beta_2 = 2\pi$ the depth b equals the wavelength and $Q/Q_{\text{slab}} = 1{\cdot}03$.

A maximum heat flux occurs at $\beta_1 \to 0$, $\beta_2 \to 0$, $\beta_2 < \beta_1 < 0$ and we find in this limit

$$Q/Q_{\text{slab}} \to 2.$$

Hence, as one would expect, the maximum heat-flux increase is for a thin ribbon of wavy film of maximum amplitude and the flux in this case is just doubled.

Therefore, for a Gerstner wave, $1 \leqslant (Q/Q_{\text{slab}}) \leqslant 2$.

This is in fact a purely geometric result. At $\beta = 0$ the surface length of the Gerstner wave is simply $\surd 2\lambda$ and therefore for a thin ribbon the depth as compared to a slab is decreased by a factor $\surd 2$. Thus the flux/unit wavelength is precisely doubled.

4. Extensions

The higher-order approximations can be determined from (3.2) and (3.3). For example, an approximation to $g_1(\beta)$, which is accurate to $O(\epsilon)$ is, from (3.6),

$$g_1(\beta) = -\epsilon L_2^{(1)} f_0(\beta),$$

and from (3.7) it is evident that $f_1 = O(\epsilon^2)$. These results can in turn be used to obtain the second-order approximation $T^2(\beta)$ to $f_0(\beta)$, [the first-order approximation $T^1(\beta) \equiv T^0(\beta)$], and improved estimates of flux could be made. Since the corrections are uniformly of order ϵ^2 and ϵ^2 is typically $O(10^{-8})$ for liquids it seems to be unnecessary to construct solutions of better than zeroth-order accuracy. It is also possible (O'Brien 1965) to construct a more general Lagrangian description which includes the Gerstner wave as a special case and which is also an accurate representation of gravity waves of moderate slope. The zeroth-order approximations to the temperature field and heat flux can be deduced in a manner entirely analogous to the method used above and one interesting result is that mentioned in the last paragraph of the introduction.

The simple approximation developed in §3 depends on the smallness of ϵ which can be written as Dk^2/ck and which is clearly the ratio between the time scale for a particle period and the time scale associated with molecular diffusion over a diameter of the largest particle orbit. It should be equally as applicable to estimating unsteady transfer of heat across the same kinds of waves provided that the time scale of a particle period is very much less than the time scale associated with the change of the average temperature of any particle in the layer. Specifically, if the average surface temperature is made time-dependent but with a large time scale relative to a particle period then the arguments of §3 will hold locally in time. The coefficients $f_n(\beta)$ and $g_n(\beta)$ will now be time-dependent with respect to the longer time scale and there will be a contribution from the time derivative on the left-hand side of (2.2) which will account for the change in the average temperature of particles.

The consequences to the zeroth-order approximation are predictable. It now must satisfy

$$T_t(\beta, t) = Dk^2 J^{-3} L_1^{(0)} T(\beta, t),$$
$$T(\beta_1, t) = T_1(t), \quad T(\beta_2, t) = T_2(t),$$
$$T(\beta, 0) \quad \text{a prescribed function.}$$

The solution of such a problem is routine. A typical example has been computed in detail (O'Brien 1965).

A similar extension of the solution is evidently possible in the event that the average temperature field of the particles is not homogeneous in the horizontal plane provided only that the length scale of any inhomogeneity is very much larger than a wavelength of the wave.

5. Conclusion

It has been shown that progressive waves on a thin layer of fluid can augment the flux of heat through the layer. The significant physical feature, for fluids in which $(Dk)/c$ is sufficiently small, is that the fluid particles maintain an almost constant temperature in their orbital motion.

By adopting a Lagrangian description of the wave-motion an accurate zeroth-order approximation to the temperature field of the wave is obtained for the case of constant average surface temperatures. The related flux results are in qualitative agreement with Nusselt's (1923) remarks concerning the role of waves in increasing the flux of heat through a liquid layer.

The problem studied here was originally suggested by Professor Walter S. Bradfield, and Mr Tore Omholt carried out many of the calculations of §§2 and 3. The author is grateful for their participation and for the numerous discussions of the problem carried on with Dr Michael Bentwich. This work was supported by National Science Foundation Grant no. 31-82.

REFERENCES

Lamb, H. 1945 *Hydrodynamics*. New York: Dover.

McAdams, W. H. 1954 *Heat Transmission*. New York: McGraw-Hill.

O'Brien, E. E. 1965 State University of New York at Stony Brook, Stony Brook, New York, *College of Engineering Rept.* no. 28.

Nusselt, W. 1923 *Z. Ver. deut. Ing.* **67**, 206.